Structural spin-glass identities from a stability property: an explicit derivation

Pierluigi Contucci [1], Cristian Giardinà [2], Claudio Giberti [3]

Abstract

In this paper a recent extension [1] of the stochastic stability property [2] is analyzed and shown to lead to the Ghirlanda Guerra identities for Gaussian spin glass models. The result is explicitly obtained by integration by parts techinque.

1 Definitions

We consider a disordered model of Ising configurations $\sigma_n = \pm 1$, $n \in \Lambda \subset \mathcal{L}$ for some subset Λ (volume $|\Lambda|$) of some infinite graph $\mathcal{L}$. We denote by Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n\in\Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. In the sequel the following definitions will be used.

1. *Hamiltonian.*

   For every $\Lambda \subset \mathcal{L}$ let $\{H_\Lambda(\sigma)\}_{\sigma\in\Sigma_\Lambda}$ be a family of $2^{|\Lambda|}$ *translation invariant (in distribution) Gaussian* random variables defined according to the general representation

$$H_\Lambda(\sigma) \;=\; -\sum_{X\subset\Lambda} J_X \sigma_X \tag{1.1}$$

   where

$$\sigma_X = \prod_{i\in X} \sigma_i \;, \tag{1.2}$$

[1] pierluigi.contucci@unibo.it, Università di Bologna, Piazza di Porta S.Donato 5 - 40127 Bologna, Italy

[2] cristian.giardina@unimore.it, Università di Modena e Reggio E., viale A. Allegri, 9 - 42121 Reggio Emilia, Italy

[3] claudio.giberti@unimore.it, Università di Modena e Reggio E., via G. Amendola 2 - Pad. Morselli- 42122 Reggio Emilia, Italy

($\sigma_\emptyset = 0$) and the J's are independent Gaussian variables with mean

$$\mathrm{Av}(J_X) = 0 \; , \tag{1.3}$$

and variance

$$\mathrm{Av}(J_X^2) = \Delta_X^2 \; . \tag{1.4}$$

2. *Average and Covariance matrix.*

The Hamiltonian $H_\Lambda(\sigma)$ has covariance matrix

$$\begin{aligned} \mathcal{C}_\Lambda(\sigma,\tau) \quad &:= \quad \mathrm{Av}\left(H_\Lambda(\sigma)H_\Lambda(\tau)\right) \\ &= \quad \sum_{X\subset\Lambda} \Delta_X^2 \sigma_X \tau_X \; . \end{aligned} \tag{1.5}$$

The two classical examples are the covariances of the Sherrington-Kirkpatrick model and the Edwards-Anderson model. A simple computation shows that the first is the square of the site overlap $\frac{1}{N}\sum_{i=1}^N \sigma_i\tau_i$ and the second is the link-overlap $\frac{1}{|\Lambda|}\sum_{|i-j|=1}\sigma_i\sigma_j\tau_i\tau_j$. Defining

$$D_\Lambda := \mathcal{C}_\Lambda(\sigma,\sigma) = \sum_{X\subset\Lambda}\Delta_X^2, \tag{1.6}$$

by the Schwarz inequality we obtain

$$|\mathcal{C}_\Lambda(\sigma,\tau)| \;\le\; \sqrt{\mathcal{C}_\Lambda(\sigma,\sigma)}\sqrt{\mathcal{C}_\Lambda(\tau,\tau)} \;=\; D_\Lambda \tag{1.7}$$

for all σ and τ.

3. *Thermodynamic Stability.*

The Hamiltonian (1.1) is thermodynamically stable if there exists a constant $\bar{c}$ such that

$$\sup_{\Lambda\subset\mathcal{L}} \frac{1}{|\Lambda|}\sum_{X\subset\Lambda}\Delta_X^2 \quad\le\quad \bar{c} \;<\; \infty \; . \tag{1.8}$$

Thanks to the relation (1.7) a thermodynamically stable model fulfills the bound

$$\mathcal{C}_\Lambda(\sigma,\tau) \quad\le\quad \bar{c}\,|\Lambda| \tag{1.9}$$

and has an order 1 normalized covariance

$$c_\Lambda(\sigma,\tau) \quad := \quad \frac{1}{|\Lambda|}\mathcal{C}_\Lambda(\sigma,\tau)\ . \tag{1.10}$$

4. *Random partition function.*

$$\mathcal{Z}_\Lambda(\beta) \ := \ \sum_{\sigma\in\,\Sigma_\Lambda} e^{\beta H_\Lambda(\sigma)}, \tag{1.11}$$

5. *Random Boltzmann-Gibbs state*

$$\omega_{\beta,\Lambda}(-) \ := \ \sum_{\sigma\in\Sigma_\Lambda}(-)\frac{e^{\beta H_\Lambda(\sigma)}}{\mathcal{Z}_\Lambda(\beta)}\ , \tag{1.12}$$

and its R-product version

$$\Omega_{\beta,\Lambda}(-) \ := \ \sum_{\sigma^{(1)},\dots,\sigma^{(R)}}(-)\,\frac{e^{\beta[H_\Lambda(\sigma^{(1)})+\cdots+H_\Lambda(\sigma^{(R)})]}}{[\mathcal{Z}_\Lambda(\beta)]^R}\ . \tag{1.13}$$

6. *Quenched overlap observables.*

For any smooth bounded function $G(c_\Lambda)$ (without loss of generality we consider $|G|\le 1$ and no assumption of permutation invariance on G is made) of the covariance matrix entries we introduce (with a small abuse of notation) the random $R\times R$ matrix of elements $\{c_{k,l}\}$ (called *generalized overlap*) and its measure $\langle - \rangle_\Lambda$ by the formula

$$\langle G(c)\rangle_\Lambda \ := \ \mathrm{Av}\left(\Omega_{\beta,\Lambda}(G(c_\Lambda))\right)\ . \tag{1.14}$$

E.g.: $G(c_\Lambda)=c_\Lambda(\sigma^{(1)},\sigma^{(2)})c_\Lambda(\sigma^{(2)},\sigma^{(3)})$

$$\langle c_{1,2}c_{2,3}\rangle_\Lambda \ = \ \mathrm{Av}\left(\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} c_\Lambda(\sigma^{(1)},\sigma^{(2)})c_\Lambda(\sigma^{(2)},\sigma^{(3)})\ \frac{e^{\beta[\sum_{i=1}^3 H_\Lambda(\sigma^{(i)})]}}{[\mathcal{Z}(\beta)]^3}\right)\ . \tag{1.15}$$

2 Standard Stochastic Stability

Given the Gaussian process $H_\Lambda(\sigma)$ of covariance $\mathcal{C}_\Lambda(\sigma,\tau)$ and an independent Gaussian process, $K_\Lambda(\sigma)$, defined by the covariance $c_\Lambda(\sigma,\tau)$, we introduce the deformed random state

$$\omega_\Lambda^{(\lambda)}(-) \ = \ \frac{\omega_{\beta,\Lambda}(-e^{\lambda K_\Lambda})}{\omega_{\beta,\Lambda}(e^{\lambda K_\Lambda})} \tag{2.16}$$

and its relative deformed quenched state

$$\langle - \rangle_\Lambda^{(\lambda)} = \mathrm{Av}\left(\Omega_\Lambda^{(\lambda)}(-)\right) , \tag{2.17}$$

where $\Omega_\Lambda^{(\lambda)}(-)$ is the R-fold product of $\omega_\Lambda^{(\lambda)}$.

Definition 2.1 Stochastic Stability [2, 4]
A Gaussian spin glass model is stochastically stable if the deformed quenched state and the original one do coincide in the thermodynamic limit:

$$\lim_{\Lambda \nearrow \mathcal{L}} \langle - \rangle_\Lambda^{(\lambda)} \;=\; \lim_{\Lambda \nearrow \mathcal{L}} \langle - \rangle_\Lambda \; . \tag{2.18}$$

Since the Hamiltonian H and the field K have a mutually rescaled distribution

$$H_\Lambda \stackrel{\mathcal{D}}{=} \sqrt{|\Lambda|} K_\Lambda \tag{2.19}$$

(where $\stackrel{\mathcal{D}}{=}$ means equality in distribution) the addition law for the Gaussian variables implies

$$\sqrt{\beta^2 + \frac{\lambda^2}{|\Lambda|}} H(\sigma) \;\stackrel{\mathcal{D}}{=}\; \beta H(\sigma) + \lambda K(\sigma) \; , \tag{2.20}$$

i.e. the deformation with a field K is equivalent to a change of the order $O(\frac{1}{|\Lambda|})$ in the temperature. The previous formula shows that the deformed measures do coincide, a part on points of discontinuity with respect to the temperature, with the original unperturbed one.

The stochastic stability property implies the vanishing, in the thermodynamic limit, of all the derivatives of the deformed state :

$$\lim_{\Lambda \nearrow \mathcal{L}} \frac{\partial^n \langle - \rangle_\Lambda^{(\lambda)}}{\partial \lambda^n} = 0 \; . \tag{2.21}$$

This formulation of the stability property implies some overlap identities. The simplest one is obtained considering: $\langle c_{1,2} \rangle_\Lambda^{(\lambda)}$. The fact that the first derivative in λ is equal to zero (in the thermodynamic limit)

$$\lim_{\Lambda \nearrow \mathcal{L}} \frac{\partial \langle c_{1,2} \rangle_\Lambda^{(\lambda)}}{\partial \lambda} |_{\lambda=0} = 0 \tag{2.22}$$

does not give information because actually for every volume one has

$$\frac{\partial \langle c_{1,2}\rangle_\Lambda^{(\lambda)}}{\partial\lambda}|_{\lambda=0} = 0. \tag{2.23}$$

This is immediately realized by defining

$$f(\lambda) = \sqrt{\beta^2 + \frac{\lambda^2}{|\Lambda|}}$$

and noticing that

$$f'(\lambda)|_{\lambda=0} = 0.$$

However the second derivative being equal to zero in the thermodynamic limit

$$\lim_{\Lambda\nearrow\mathcal{L}} \frac{\partial^2 \langle c_{1,2}\rangle_\Lambda^{(\lambda)}}{\partial\lambda^2}|_{\lambda=0} = 0 \tag{2.24}$$

does give information, since

$$f''(\lambda)|_{\lambda=0} = \frac{1}{\beta|\Lambda|}.$$

Indeed an explicit computation of

$$\frac{\partial^2 \langle c_{1,2}\rangle_\Lambda^{(\lambda)}}{\partial\lambda^2}|_{\lambda=0} \tag{2.25}$$

which uses integration by parts (see the next section) gives the first Aizenman-Contucci polynomial:

$$\lim_{\Lambda\nearrow\mathcal{L}} \langle c_{1,2}^2 - 4c_{1,2}c_{2,3} + 3c_{1,2}c_{3,4}\rangle_\Lambda = 0. \tag{2.26}$$

Besides Stochastic Stability, there is another mechanism which generates identities. This is a very basic principle of statistical mechanics, i.e. the vanishing of the fluctuation of the energy per particle (self averaging): at increasing volumes the energy per particle approaches a constant with respect to the equilibrium measure. The consequence of the self averaging is a family of relations called Ghirlanda-Guerra identities [3, 5].

Theorem 1 (Ghirlanda-Guerra Identities) *For a bounded function v of the generalized overlaps $\{c_{i,j}\}$ (with $i,j \in \{1,...,s\}$) the quantity $\delta_\Lambda(\beta)$ defined by:*

$$\langle v\, c_{1,s+1}\rangle_\Lambda = \frac{1}{s}\langle v\rangle_\Lambda\, \langle c_{1,2}\rangle_\Lambda + \frac{1}{s}\sum_{j=2}^{s}\langle v\, c_{1,j}\rangle_\Lambda + \delta_\Lambda(\beta) \tag{2.27}$$

goes to zero in $\beta-$average and in the thermodynamic limit: $\Lambda \nearrow \mathcal{L}$.

3 A perturbed state

Let us introduce a new state which, unlike (2.17), does not involve an indipendent Gaussian process as a perturbing therm. In fact in this case we perturb the state through a small deformation $\Delta(\lambda)H_\Lambda$ of the *same* Hamiltonian which defines the model:

$$\langle\langle - \rangle\rangle_\Lambda^{(\lambda)} := \frac{\mathrm{Av}\left(\omega_{\beta,\Lambda}(-\ e^{\Delta(\lambda)H_\Lambda})\right)}{\mathrm{Av}\left(\omega_{\beta,\Lambda}(e^{\Delta(\lambda)H_\Lambda})\right)}, \tag{3.28}$$

where $\Delta(\lambda) \equiv \Delta_\Lambda(\lambda)$ is any function satisfying

$$\Delta_\Lambda(0) = 0, \quad \Delta_\Lambda(\lambda) \to 0 \text{ as } |\Lambda| \to \infty, \quad \Delta'_\Lambda(0) = a/|\Lambda| \tag{3.29}$$

(with a positive constant), e.g. $\Delta_\Lambda(\lambda) = \lambda/|\Lambda|$. Obviously $\langle\langle - \rangle\rangle_\Lambda^{(0)} = \langle - \rangle_\Lambda$. The explicit the expression of (3.28) reads

$$\langle\langle f \rangle\rangle_\Lambda^{(\lambda)} = \frac{\mathrm{Av}\left(\frac{\sum_\sigma f(\sigma) e^{(\beta+\Delta(\lambda))H_\Lambda(\sigma)}}{\sum_\sigma e^{\beta H_\Lambda(\sigma)}}\right)}{\mathrm{Av}\left(\frac{\sum_\sigma e^{(\beta+\Delta(\lambda))H_\Lambda(\sigma)}}{\sum_\sigma e^{\beta H_\Lambda(\sigma)}}\right)}, \tag{3.30}$$

where f is a function of the spin configurations. It is useful to define a symbol for denoting the random measure $\omega_\Lambda(-\ e^{\Delta(\lambda)H_\Lambda})$ introduced in (3.28) and its R-fold products:

$$\phi_\Lambda^{(\lambda)}(-) := \omega_\Lambda(-\ e^{\Delta(\lambda)H_\Lambda}) = \sum_\sigma (-) \frac{e^{g(\lambda)H_\Lambda(\sigma)}}{\mathcal{Z}_\Lambda(\beta)}\ ,\ \Phi_\Lambda^{(\lambda,\ldots,\lambda)}(-) := \sum_{\sigma^{(1)},\ldots,\sigma^{(R)}} (-) \prod_{r=1}^R \frac{e^{g(\lambda)H_\Lambda(\sigma^{(r)})}}{\mathcal{Z}_\Lambda(\beta)}\ , \tag{3.31}$$

where

$$g(\lambda) = \beta + \Delta_\Lambda(\lambda) \tag{3.32}$$

and $\mathcal{Z}_\Lambda(\beta)$ is defined in (1.11). Obviously $\phi^{(0)} = \omega_\Lambda$ while $\Phi^{(0,0)}$ is identical to Ω_Λ with 2 copies, $\Phi^{(0,0,0)}$ is Ω_Λ with 3 copies etc... and, for instance, $\Phi^{(\lambda,0)}$ is the random product state in which only the first copy is perturbed.

The quenched versions of the previous measures are:

$$[-]_\Lambda^{(\lambda)} := \mathrm{Av}\left(\phi_\Lambda^{(\lambda)}(-)\right), \quad [-]_\Lambda^{(\lambda,\ldots,\lambda)} := \mathrm{Av}\left(\Phi_\Lambda^{(\lambda,\ldots,\lambda)}(-)\right), \tag{3.33}$$

thus $\langle\langle - \rangle\rangle_\Lambda^{(\lambda)} = \frac{[-]_\Lambda^{(\lambda)}}{[1]_\Lambda^{(\lambda)}}$. The same perturbation of (3.28) applied to R copies of the system, yelds the measure on the replicated system for which we retain the same symbol used for the 1-copy version:

$$\langle\langle - \rangle\rangle_\Lambda^{(\lambda)} = \frac{[-]_\Lambda^{(\lambda,\dots,\lambda)}}{[1]_\Lambda^{(\lambda,\dots,\lambda)}}. \tag{3.34}$$

Remark: *We observe that while the stochastic stability perturbation (2.17), as much as the standard perturbation for deterministic system, amounts to a small temperature shift, the newly introduced perturbation cannot be reduced to just a small temperature change but it also involves a small change in the disorder. Indeed, we can rewrite (3.28) as follows*

$$\langle\langle - \rangle\rangle_\Lambda^{(\lambda)} = \frac{\mathrm{Av}\left(Q_{\beta,\lambda} \cdot \omega_{\beta+\Delta(\lambda),\Lambda}(-)\right)}{\mathrm{Av}\left(Q_{\beta,\lambda}\right)}. \tag{3.35}$$

where

$$Q_{\beta,\lambda} := \frac{\mathcal{Z}_\Lambda(\beta+\Delta(\lambda))}{\mathcal{Z}_\Lambda(\beta)}. \tag{3.36}$$

Therefore, defining a new disorder average

$$\mathrm{Av}^{(\lambda)}(-) := \frac{\mathrm{Av}\left(Q_{\beta,\lambda} \cdot -\right)}{\mathrm{Av}\left(Q_{\beta,\lambda}\right)}, \tag{3.37}$$

we have:

$$\langle\langle - \rangle\rangle_\Lambda^{(\lambda)} = \mathrm{Av}^{(\lambda)}\left(\omega_{\beta+\Delta(\lambda),\Lambda}(-)\right), \tag{3.38}$$

which shows clearly that the new state is the composition of a themperature shift with a suitable deformation of the disorder.

Going through the same steps of section 2, we want the explore the content of the perturbation (3.28) computing the derivatives of $\langle\langle c_{1,2} \rangle\rangle_\Lambda^{(\lambda)}$; since we required that $g'(\lambda)|_{\lambda=0} \neq 0$, it will be enough to consider the first derivative. The computation requires the following important lemma [6]:

Lemma 1 (Gaussian integration by parts) *Let $\{x_1, x_2, \dots, x_n\}$ a family of Gaussian random variables and $\psi(z_1, \dots, z_n)$ a smooth function of at most polynomial growth. Then*

$$\mathrm{Av}\left(x_i \psi(x_1, \dots, x_n)\right) = \sum_{j=1}^{n} \mathrm{Av}\left(x_i x_j\right) \mathrm{Av}\left(\frac{\partial \psi(x_1, \dots, x_n)}{\partial x_j}\right). \tag{3.39}$$

$\square$

The first result of the paper is the following:

Theorem 2 *Considering the perturbed state (3.28) with perturbation $\Delta_\Lambda(\lambda)$ satisfying (3.29), we have*

$$\left.\frac{\partial \langle\langle c_{1,2}\rangle\rangle_\Lambda^{(\lambda)}}{\partial\lambda}\right|_{\lambda=0} = 2a\beta\left(\langle c_{1,2}^2\rangle_\Lambda - 2\langle c_{1,2}c_{2,3}\rangle_\Lambda + \langle c_{1,2}\rangle_\Lambda^2\right).$$

Proof:

Since the gaussian integration by parts formula involves the covariance of the hamiltonian family, it is convenient to write

$$\langle\langle c_{1,2}\rangle\rangle_\Lambda^{(\lambda)} = \frac{1}{|\Lambda|}\frac{A_1(\lambda)}{B_1(\lambda)} \tag{3.40}$$

with

$$A_1(\lambda) = [C_\Lambda]_\Lambda^{(\lambda)} = \mathrm{Av}\left(\sum_{\sigma,\tau} C_{\sigma,\tau}\frac{e^{g(\lambda)H_\Lambda(\sigma)}e^{g(\lambda)H_\Lambda(\tau)}}{\mathcal{Z}_\Lambda(\beta)^2}\right) \tag{3.41}$$

where $C_{\sigma,\tau} := \mathcal{C}_\Lambda(\sigma,\tau)$ are the elements of the covariance matrix C_Λ given in (1.5) (extensive quantities), and

$$B_1(\lambda) = [1]_\Lambda^{(\lambda)} = \mathrm{Av}\left(\sum_{\sigma,\tau}\frac{e^{g(\lambda)H_\Lambda(\sigma)}e^{g(\lambda)H_\Lambda(\tau)}}{\mathcal{Z}_\Lambda(\beta)^2}\right). \tag{3.42}$$

Let us compute the derivative of (3.40) starting from:

$$\frac{dA_1(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\sigma,\tau} C_{\sigma,\tau}\left(H_\Lambda(\sigma)+H_\Lambda(\tau)\right)\frac{e^{g(\lambda)H_\Lambda(\sigma)}e^{g(\lambda)H_\Lambda(\tau)}}{\mathcal{Z}_\Lambda(\beta)^2}\right). \tag{3.43}$$

Applying the integration by parts formula and recalling (1.5), we have:

$$\begin{aligned}
&\mathrm{Av}\left(H_\Lambda(\sigma)\frac{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))}{\mathcal{Z}_\Lambda(\beta)^2}\right) = \sum_\eta C_{\sigma,\eta}\mathrm{Av}\left(\frac{\partial}{\partial H_\Lambda(\eta)}\;\frac{e^{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))}}{\mathcal{Z}_\Lambda(\beta)^2}\right)\\
= &\sum_\eta C_{\sigma,\eta}\,\mathrm{Av}\left(\left[g(\lambda)(\delta_{\sigma,\eta}+\delta_{\tau,\eta}) - 2\beta\frac{e^{\beta H_\Lambda(\eta)}}{\mathcal{Z}_\Lambda(\beta)}\right]\frac{e^{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))}}{\mathcal{Z}_\Lambda(\beta)^2}\right),
\end{aligned} \tag{3.44}$$

where $\delta_{\sigma,\eta}$ is the Kronecker delta function. Multiplying the last therm by $C_{\sigma,\tau}$ and summing over the configurations, we have

$$\begin{aligned}
&g(\lambda)\mathrm{Av}\left(\sum_{\sigma,\tau} C_{\sigma,\tau}C_{\sigma,\sigma}\frac{e^{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))}}{\mathcal{Z}_\Lambda(\beta)^2}\right) + g(\lambda)\mathrm{Av}\left(\sum_{\sigma,\tau} C_{\sigma,\tau}^2\frac{e^{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))}}{\mathcal{Z}_\Lambda(\beta)^2}\right)\\
- &2\beta\mathrm{Av}\left(\sum_{\sigma,\tau,\eta} C_{\sigma,\tau}C_{\sigma,\eta}\frac{e^{g(\lambda)(H_\Lambda(\sigma)+H_\Lambda(\tau))+\beta H_\Lambda(\eta))}}{\mathcal{Z}_\Lambda(\beta)^3}\right)\\
= &D_\Lambda g(\lambda)[C_{1,2}]_\Lambda^{(\lambda,\lambda)} + g(\lambda)[C_{1,2}^2]_\Lambda^{(\lambda,\lambda)} - 2\beta[C_{1,2}C_{2,3}]_\Lambda^{(\lambda,\lambda,0)}
\end{aligned} \tag{3.45}$$

where D_Λ is defined in (1.6). Thus

$$\frac{dA_1(\lambda)}{d\lambda} = 2D_\Lambda g(\lambda)g'(\lambda)[C_{1,2}]_\Lambda^{(\lambda,\lambda)} + 2g(\lambda)g'(\lambda)[C_{1,2}^2]_\Lambda^{(\lambda,\lambda)} - 4\beta g'(\lambda)[C_{1,2}C_{2,3}]_\Lambda^{(\lambda,\lambda,0)}. \quad (3.46)$$

The derivative of $B_1(\lambda)$

$$\frac{dB_1(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\sigma,\tau}(H_\Lambda(\sigma)+H_\Lambda(\tau))\frac{e^{g(\lambda)H_\Lambda(\sigma)}e^{g(\lambda)H_\Lambda(\tau)}}{\mathcal{Z}_\Lambda(\beta)^2}\right). \quad (3.47)$$

can be obtained from the previous computation by formally substituting $C_{\sigma,\tau}$ with 1:

$$\frac{dB_1(\lambda)}{d\lambda} = 2D_\Lambda g(\lambda)g'(\lambda)\mathrm{Av}\left(m(\lambda)^2\right)+2g(\lambda)g'(\lambda)[C_{1,2}]_\Lambda^{(\lambda,\lambda)}-4\beta g'(\lambda)\mathrm{Av}\left(m(\lambda)\phi^{(\lambda,0)}(C_{1,2})\right). \quad (3.48)$$

where $m(\lambda)=\phi^{(\lambda)}(1)$, $(m(0)=1)$. Computing the derivatives in zero and recalling (1.10), we find $(d_\Lambda = D_\Lambda/|\Lambda|)$:

$$\left.\frac{dA_1(\lambda)}{d\lambda}\right|_{\lambda=0} = 2\beta a|\Lambda|\left(d_\Lambda\langle c_{1,2}\rangle_\Lambda + \langle c_{1,2}^2\rangle_\Lambda - 2\langle c_{1,2}c_{2,3}\rangle_\Lambda\right) \quad (3.49)$$

and

$$\left.\frac{dB_1(\lambda)}{d\lambda}\right|_{\lambda=0} = 2\beta a(d_\Lambda - \langle c_{1,2}\rangle_\Lambda). \quad (3.50)$$

Since

$$\frac{d}{d\lambda}\left(\frac{A_1(\lambda)}{B_1(\lambda)}\right) = \frac{A_1'(\lambda)B_1(\lambda)-A_1(\lambda)B_1'(\lambda)}{B_1(\lambda)^2}, \quad (3.51)$$

using (3.49),(3.50), and the fact that $A_1(0)=|\Lambda|\langle c_{1,2}\rangle_\Lambda$ and $B_1(0)=1$, we immediately deduce that:

$$\left.\frac{\partial\langle\langle c_{1,2}\rangle\rangle_\Lambda^{(\lambda)}}{\partial\lambda}\right|_{\lambda=0} = \frac{1}{|\Lambda|}\frac{d}{d\lambda}\left.\left(\frac{A_1(\lambda)}{B_1(\lambda)}\right)\right|_{\lambda=0} = 2\beta a\left(\langle c_{1,2}^2\rangle_\Lambda - 2\langle c_{1,2}c_{2,3}\rangle_\Lambda + \langle c_{1,2}\rangle_\Lambda^2\right). \quad (3.52)$$

$\square$

The same computation can be extended to a generic function v of the overlaps of s copies (the previous theorem corresponds to the case $v = c_{1,2}$). Here we denote by c the collection of all the entries $c = \{c_{i,j}\}_{i,j=1\ldots,s}$. Recalling the definition (3.34) of the deformed product state, we have:

$$\langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)} = \frac{\mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(s)}} v(c)\frac{e^{g(\lambda)(H_\Lambda(\sigma^{(1)})+\ldots+H_\Lambda(\sigma^{(s)}))}}{\mathcal{Z}(\beta)^s}\right)}{\mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(s)}}\frac{e^{g(\lambda)(H_\Lambda(\sigma^{(1)})+\ldots+H_\Lambda(\sigma^{(s)}))}}{\mathcal{Z}(\beta)^s}\right)} \quad (3.53)$$

where $\sigma^{(j)}$ represents the generic configuration of the $j-$th copy of the system.

Theorem 3 (Deformation of s copies) *Let v be a function of the overlaps of s copies, then for the deformed average (3.53) with perturbation $\Delta_\Lambda(\lambda)$ satisfying (3.29), we have*

$$\left.\frac{\partial\langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)}}{\partial\lambda}\right|_{\lambda=0} = a\beta\left(\sum_{\substack{\ell,k=1\\ \ell\neq k}}^{s}\langle v(c)\; c_{\ell,k}\rangle_\Lambda + s\langle v(c)\rangle_\Lambda\langle c_{1,2}\rangle_\Lambda - s\sum_{\ell=1}^{s}\langle v(c)\; c_{\ell,s+1}\rangle_\Lambda\right) \quad (3.54)$$

Proof:We now define

$$\langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)} \;=\; \frac{A_2(\lambda)}{B_2(\lambda)} \quad (3.55)$$

and, for the sake of notation

$$S(\hat\sigma) = \sum_{j=1}^{s} H_\Lambda(\sigma^{(j)}), \quad (3.56)$$

where $\hat\sigma = (\sigma^{(1)},\ldots,\sigma^{(s)}) \in \Sigma_\Lambda^s$ is the generic configuration of the product system. The derivative of $B_2(\lambda)$ is

$$\frac{dB_2(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\hat\sigma} S(\hat\sigma)\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right) = g'(\lambda)\sum_{k=1}^{s}\mathrm{Av}\left(\sum_{\hat\sigma} H_\Lambda(\sigma^{(k)})\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right) \quad (3.57)$$

The computation of the summand in (3.57) goes parallel to that of (3.61), resulting in

$$\mathrm{Av}\left(H_\Lambda(\sigma^{(k)})\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right) = \sum_\eta C_{\sigma^{(k)},\eta}\;\mathrm{Av}\left(\left[g(\lambda)\left(\sum_{j=1}^{s}\delta_{\sigma^{(j)},\eta}\right) - 2\beta\frac{e^{\beta H_\Lambda(\eta)}}{\mathcal{Z}_\Lambda(\beta)}\right]\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right). \quad (3.58)$$

Summing over $\hat\sigma$ and k, we obtain

$$g(\lambda)\sum_{j,k=1}^{s}\mathrm{Av}\left(\sum_{\hat\sigma} C_{\sigma^{(k)},\sigma^{(j)}}\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right) - s\beta\sum_{k=1}^{s}\mathrm{Av}\left(\sum_{\hat\sigma,\eta} C_{\sigma^{(k)},\eta}\frac{e^{g(\lambda)S(\hat\sigma)}e^{\beta H_\Lambda(\eta)}}{\mathcal{Z}_\Lambda(\beta)^{s+1}}\right). \quad (3.59)$$

Thus, recalling the notations introduced in (3.33), we obtain

$$\frac{dB_2(\lambda)}{d\lambda} = g(\lambda)g'(\lambda)\sum_{j,k=1}^{s}[C_{j,k}]_\Lambda^{(\lambda,\ldots,\lambda)} - s\beta g'(\lambda)\sum_{j=1}^{s}[C_{j,s+1}]_\Lambda^{(\lambda,\ldots,\lambda,0)}. \quad (3.60)$$

The derivative of $A_2(\lambda)$ is

$$\frac{dA_2(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\hat\sigma} v(c)S(\hat\sigma)\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right) = g'(\lambda)\sum_{k=1}^{s}\mathrm{Av}\left(\sum_{\hat\sigma} v(c)H_\Lambda(\sigma^{(k)})\frac{e^{g(\lambda)S(\hat\sigma)}}{\mathcal{Z}_\Lambda(\beta)^s}\right), \quad (3.61)$$

therefore it can be computed formally by inserting $v(c)$ in (3.57):

$$g(\lambda)\sum_{j,k=1}^{s}\mathrm{Av}\left(\sum_{\hat{\sigma}} v(c)C_{\sigma^{(k)},\sigma^{(j)}}\frac{e^{g(\lambda)S(\hat{\sigma})}}{\mathcal{Z}_\Lambda(\beta)^s}\right)-s\beta\sum_{k=1}^{s}\mathrm{Av}\left(\sum_{\hat{\sigma},\,\eta} v(c)C_{\sigma^{(k)},\eta}\frac{e^{g(\lambda)S(\hat{\sigma})}e^{\beta H_\Lambda(\eta)}}{\mathcal{Z}_\Lambda(\beta)^{s+1}}\right) \quad (3.62)$$

The final result is

$$\frac{dA_2(\lambda)}{d\lambda}=g(\lambda)g'(\lambda)\sum_{j,k=1}^{s}[v(c)C_{j,k}]_\Lambda^{(\lambda,\dots,\lambda)}-s\beta g'(\lambda)\sum_{j=1}^{s}[v(c)C_{j,s+1}]_\Lambda^{(\lambda,\dots,\lambda,0)}. \quad (3.63)$$

Computing the derivatives in $\lambda=0$, we obtain

$$\begin{aligned}\left.\frac{dB_2(\lambda)}{d\lambda}\right|_{\lambda=0} &= a\beta\sum_{j,k=1}^{s}\langle c_{j,k}\rangle_\Lambda - sa\beta\sum_{j=1}^{s}\langle c_{j,s+1}\rangle_\Lambda \\ &= a\beta\left((s^2-s)\langle c_{1,2}\rangle_\Lambda + sd_\Lambda - s^2\langle c_{1,2}\rangle_\Lambda\right) = a\beta s\left(d_\Lambda-\langle c_{1,2}\rangle_\Lambda\right) \quad (3.64)\end{aligned}$$

since $\langle c_{j,k}\rangle$ is independent of the replica indices and, being the self-overlap a constant $c_{\sigma,\sigma}=d_\Lambda$, we have also $\langle c_{k,k}\rangle=d_\Lambda$ ($d_\Lambda=D_\Lambda/|\Lambda|$). For the same reason we can also write:

$$\begin{aligned}\left.\frac{dA_2(\lambda)}{d\lambda}\right|_{\lambda=0} &= a\beta\sum_{j,k=1}^{s}\langle v(c)c_{j,k}\rangle_\Lambda - sa\beta\sum_{j=1}^{s}\langle v(c)c_{j,s+1}\rangle_\Lambda \\ &= a\beta\left(\sum_{\substack{j,k=1\\ j\neq k}}^{s}\langle v(c)c_{j,k}\rangle_\Lambda + sd_\Lambda\langle v(c)\rangle_\Lambda - sa\beta\sum_{j=1}^{s}\langle v(c)c_{j,s+1}\rangle_\Lambda\right). \quad (3.65)\end{aligned}$$

The proof is completed recalling that $A_2(0)=\langle v(c)\rangle_\Lambda$ and $B_2(0)=1$.

□

The previous result can be further simplyfied assuming that the function $v(c)$ be invariant with respect the permutation of the replicas. In fact in that case the therm in (3.54) is

$$\left.\frac{\partial\langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)}}{\partial\lambda}\right|_{\lambda=0}=a\beta s\left(\sum_{k=2}^{s}\langle v(c)\ c_{1,k}\rangle_\Lambda+\langle v(c)\rangle_\Lambda\langle c_{1,2}\rangle_\Lambda-s\langle v(c)\ c_{\ell,s+1}\rangle_\Lambda\right) \quad (3.66)$$

Relaxing the invariance hypothesis on $v(c)$, we can obtain the same result perturbing *only one* replica. Without loss of generality, we assume to perturb the first copy:

$$\langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)_1} = \frac{\mathrm{Av}\left(\sum_{\sigma^{(1)},\tilde{\sigma}} v(c)\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta T_\Lambda(\tilde{\sigma})}}{\mathcal{Z}(\beta)^s}\right)}{\mathrm{Av}\left(\sum_{\sigma^{(1)},\tilde{\sigma}}\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta T_\Lambda(\tilde{\sigma})}}{\mathcal{Z}(\beta)^s}\right)} \quad (3.67)$$

where $\tilde{\sigma} = (\sigma^{(2)}, \dots, \sigma^{(s)})$ and

$$T_\Lambda(\tilde{\sigma}) = \sum_{j=2}^{s} H_\Lambda(\sigma^{(j)}). \tag{3.68}$$

Then we can state the following

Theorem 4 (Deformation of 1 copy) *Let v be a funcion of the overlaps of s copies, then for the deformed average (3.67) with perturbation $\Delta_\Lambda(\lambda)$ satisfying (3.29), we have*

$$\left.\frac{\partial \langle\langle v(c)\rangle\rangle_\Lambda^{(\lambda)_1}}{\partial\lambda}\right|_{\lambda=0} = a\beta\left(\sum_{k=2}^{s}\langle v(c)\ c_{1,k}\rangle_\Lambda + \langle v(c)\rangle_\Lambda\langle c_{1,2}\rangle_\Lambda - s\langle v(c)\ c_{\ell,s+1}\rangle_\Lambda\right) \tag{3.69}$$

Proof:

Let us denote with $A_3(\lambda)$ and $B_3(\lambda)$ the numerator and denominator of (3.67). Thus,

$$\frac{dB_3(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\sigma^{(1)},\,\tilde{\sigma}} H_\Lambda(\sigma^{(1)})\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta T_\Lambda(\tilde{\sigma})}}{\mathcal{Z}_\Lambda(\beta)^s}\right) \tag{3.70}$$

which, applying the integration by parts lemma, can be rewritten as:

$$\begin{aligned}
\frac{dB_3(\lambda)}{d\lambda} &= g'(\lambda)\mathrm{Av}\left(\sum_{\sigma^{(1)},\,\tilde{\sigma}}\sum_{\eta} C_{\sigma^{(1)},\eta}\left(g(\lambda)\delta_{\sigma^{(1)},\eta} + \beta\sum_{j=2}^{s}\delta_{\sigma^{(j)},\eta}\right)\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta T_\Lambda(\tilde{\sigma})}}{\mathcal{Z}_\Lambda(\beta)^s}\right)\\
&- s\beta\mathrm{Av}\left(\sum_{\sigma^{(1)},\,\tilde{\sigma}}\sum_{\eta} C_{\sigma^{(1)},\eta}\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta(T_\Lambda(\tilde{\sigma})+H_\Lambda(\eta))}}{\mathcal{Z}_\Lambda(\beta)^{s+1}}\right)\\
&= D_\Lambda g(\lambda)g'(\lambda)\,[1]_\Lambda^{(\lambda,0,\dots,0)} - sa\beta g'(\lambda)\,[C_{1,s+1}]_\Lambda^{(\lambda,0,\dots,0)} + \beta g'(\lambda)\sum_{j=2}^{s}[C_{1,j}]_\Lambda^{(\lambda,0,\dots,0)}\,.
\end{aligned} \tag{3.71}$$

Computing the derivative in $\lambda = 0$ and recalling that $\langle c_{i,j}\rangle_\Lambda$ is independent of the replica labels, we have:

$$\left.\frac{dB_3(\lambda)}{d\lambda}\right|_{\lambda=0} = d_\Lambda a\beta - sa\beta\langle c_{1,2}\rangle_\Lambda + (s-1)a\beta\langle c_{1,2}\rangle_\Lambda = d_\Lambda a\beta - a\beta\langle c_{1,2}\rangle_\Lambda. \tag{3.72}$$

The derivative of A_3

$$\frac{dA_3(\lambda)}{d\lambda} = g'(\lambda)\mathrm{Av}\left(\sum_{\sigma^{(1)}\tilde{\sigma}} v(c)H_\Lambda(\sigma^{(1)})\frac{e^{g(\lambda)H_\Lambda(\sigma^{(1)})+\beta T_\Lambda(\tilde{\sigma})}}{\mathcal{Z}_\Lambda(\beta)^s}\right) \tag{3.73}$$

is computed inserting $v(c)$ in (3.71) :

$$\frac{dA_3(\lambda)}{d\lambda} = D_\Lambda g(\lambda) g'(\lambda) \left[v(c)\right]_\Lambda^{(\lambda,0,\ldots,0)} - sa\beta g'(\lambda) \left[v(c) C_{1,s+1}\right]_\Lambda^{(\lambda,0,\ldots,0)} + \beta g'(\lambda) \sum_{j=2}^{s} \left[v(c) C_{1,j}\right]_\Lambda^{(\lambda,0,\ldots,0)} \tag{3.74}$$

and

$$\left.\frac{dA_3(\lambda)}{d\lambda}\right|_{\lambda=0} = d_\Lambda a\beta \langle v(c)\rangle_\Lambda - sa\beta \langle v(c) c_{1,s+1}\rangle_\Lambda + a\beta \sum_{j=2}^{s} \langle v(c) c_{1,j}\rangle_\Lambda . \tag{3.75}$$

The result is obtained combining (3.74) and (3.76) to form the derivative of $A_3(\lambda)/B_3(\lambda)$.

□

We conclude discussing briefly the stability of the new deformation. Rephrasing the definition of Sthocastic Stability, we should claim that the new state is stable if :

$$\lim_{\Lambda \nearrow \mathcal{L}} \langle\langle - \rangle\rangle_\Lambda^{(\lambda)} \;=\; \lim_{\Lambda \nearrow \mathcal{L}} \langle - \rangle_\Lambda . \tag{3.76}$$

We plan to study this strong form of asymptotic equivalence between the two states in a forthcoming paper. Here, we can make the weaker statement that the two measures coincide (for large volumes) in the first order of the perturbation parameter λ. In fact the previous theorems imply that the perturbed state, either with 1 or s deformed copies, satisfies the following realtion:

$$\langle\langle v(c) \rangle\rangle_\Lambda^{(\lambda)} - \langle v(c)\rangle_\Lambda = a_1 \, \mathcal{G}_\Lambda(v(c), s) \, \lambda + h.o.t \, , \tag{3.77}$$

where a_1 is a constant and $\mathcal{G}_\Lambda(v(c), s)$ any of the expressions involved in Theorems 2,3 or 4, e.g.:

$$\mathcal{G}_\Lambda(v(c), s) = \sum_{k=2}^{s} \langle v(c) \; c_{1,k}\rangle_\Lambda + \langle v(c)\rangle_\Lambda \langle c_{1,2}\rangle_\Lambda - s\langle v(c) \; c_{\ell,s+1}\rangle_\Lambda . \tag{3.78}$$

Thus, from theorem 1 it follows that, at the first order in λ, $\langle\langle v(c) \rangle\rangle_\Lambda^{(\lambda)} - \langle v(c)\rangle_\Lambda$ goes to zero, in $\beta-$average and in the thermodynamic limit.

Acknowledgments We acknowledge financial support from the following sources: STRATEGIC RESEARCH GRANT (University of Bologna), INTERNATIONAL RESEARCH PROJECTS (Fondazione Cassa di Risparmio, Modena) and FIRB-FUTURO IN RICERCA PROJECT RBFR10N90W "Stochastic processes in interacting particle systems" (Italian Ministry of University and Research).

References

[1] P.Contucci, C.Giardinà, C.Giberti, "Stability of the spin glass phase under perturbations", EPL. **96**, 17003 (2011)

[2] M.Aizenman, P.Contucci, "On the Stability of the Quenched state in Mean Field Spin Glass Models", Journal of Statistical Physics, Vol. **92**, N. 5/6, 765-783, (1998).

[3] S. Ghirlanda, F. Guerra, "General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity", J. Phys. A: Math. Gen., Vol. **31**, 9149-9155 (1998).

[4] P. Contucci, C. Giardinà, "Spin-Glass Stochastic Stability: a Rigorous Proof" Annales Henri Poincare Vol. **6**, No. 5, 915 - 923 (2005).

[5] P. Contucci, C. Giardinà, "The Ghirlanda-Guerra identities", Journ. Stat. Phys. Vol. **126**, 917-931 (2007).

[6] M. Talagrand, "Spin glasses: a challenge for mathematicians", Springer (2003).